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Chaos Theory and Military Applications

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In recent times, it seems as though terrorism has erupted worldwide. The acts of terrorism have increased and are no longer perpetrated by countries, but rather by various organizations located throughout the world. The mechanisms being used for these attacks have changed over the past few years. Who would ever have thought that human beings would strap bombs to themselves to hit targets more effectively, or that airplanes would be used as missiles? The escalation of terrorism has brought Chaos to many areas of the world. As a mathematics student in the Pell Scholars Honors Program, which prepares students “to seek peace and justice in the world,” I am interested in the military’s use of mathematical Chaos theory to help restore peace in the world. This paper will review the fundamentals of Chaos theory, military systems that are Chaotic, Chaos in non-physical systems, Chaos control, and the applications of Chaos theory for the military.

In his paper Chaos theory: The Essentials for Military Applications, Glenn James capitalizes the word “chaos” throughout his paper when discussing mathematical chaos. This method will be used throughout this paper as well to distinguish mathematical chaos from social chaos.

In order to help understand Chaos theory, the logistic map, a difference equation developed by Robert May, will be used. The logistic map shows the size of an animal population at the beginning of a year (x_{new}) based on the size of the population at the beginning of the previous year (x_{old}):

$$x_{new} = \lambda\, x_{old}\,(1 - x_{old})$$

$$= \lambda\, x_{old} - \lambda\, x^2_{old}$$

Since this equation has an exponent greater than one, it is **nonlinear**, which is one of the properties of the difference equation for chaotic behavior to occur within a system. The λ of this equation represents a constant. Values for this parameter range from 1 to 4. The animal population is called a **dynamical system** because the logistic equation measures the size of the population at regular time intervals.

For this difference equation to generate the size of animal populations, there must be an initial value x_0, which represents the beginning size of the population. Let $f(x) = \lambda x - \lambda x^2$. Define a sequence $x_0, x_1, x_2, \ldots, x_n \ldots$ using this function where $x_{n+1} = f(x_n)$. The generated sequence, $x_0, x_1, x_2, \ldots, x_n \ldots$, is called the **orbit of x_0 under $f(x)$** $= \lambda x - \lambda x^2$.

Finding the general characteristics of this function will aid in understanding the function better. Consider first the maximum point of the function $f(x) = - \lambda x^2 + \lambda x$. In order to find the maximum, we must take the derivative of the function and then set it to zero and solve for x.

$$f(x) = - \lambda x^2 + \lambda x$$

$$f'(x) = -2 \lambda x + \lambda$$

$$0 = \lambda(-2x + 1)$$

$$\lambda = 0 \text{ or } -2x + 1 = 0. \text{ Now } 1 \leq \lambda \leq 4.$$

Solving the second equation gives $1 = 2x$

$$(1/2) = x$$

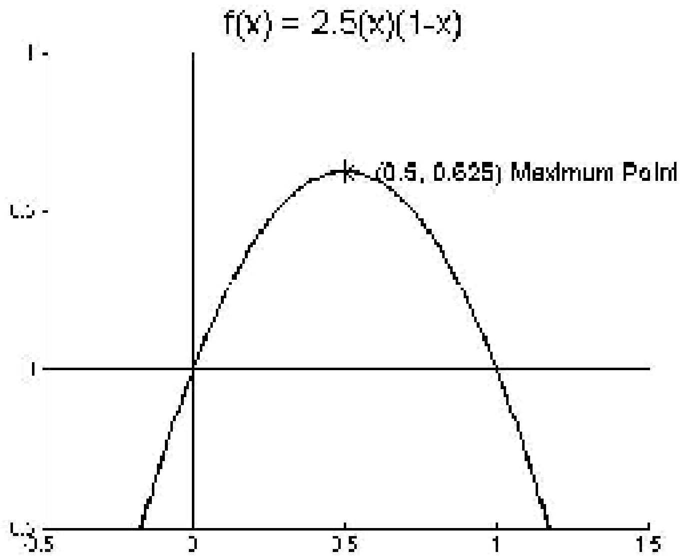
Now substitute the value of x back into the function f(x).

$$f(1/2) = - \lambda(1/2)^2 + \lambda(1/2)$$

$$f(1/2) = - (\lambda/4) + (\lambda/2)$$

$$f(1/2) = (\lambda/4)$$

Therefore, the maximum point is (1/2, λ /4). For $\lambda = 2.5$, the maximum point of $f(x) = 2.5x - 2.5x^2$ is (1/2, 5/8) = (0.5, 0.625).



The fixed points for $f(x) = \lambda x (1-x)$ are $x = 1 - (1/\lambda)$ and $x = 0$. These points were found by solving $f(x) = x$ for the variable x.

$$f(x) = x$$
$$\lambda x (1-x) = x$$
$$\lambda x - \lambda x^2 = x$$
$$\lambda x^2 - \lambda x + x = 0$$
$$x (\lambda x - \lambda + 1) = 0$$

Therefore, x = 0 is one of the fixed points.

$$\lambda x - \lambda + 1 = 0$$
$$\lambda x = \lambda - 1$$

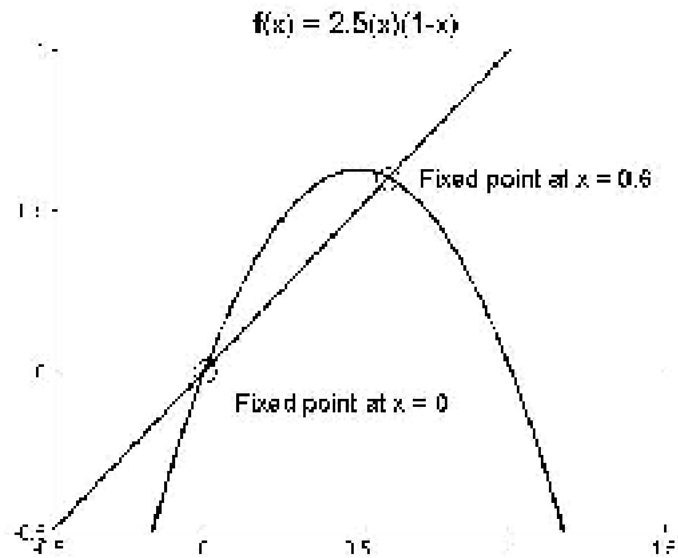
Therefore, $x = 1 - (1/\lambda)$ is the second fixed point.

Consider our example of the generating function using $\lambda = 2.5$. One of the fixed points is $x = 0$. By substituting the value of 2.5 for λ into the equation $x = 1 - (1/\lambda)$, we obtain

$$x = 1 - (1/2.5)$$

$$x = 0.6$$

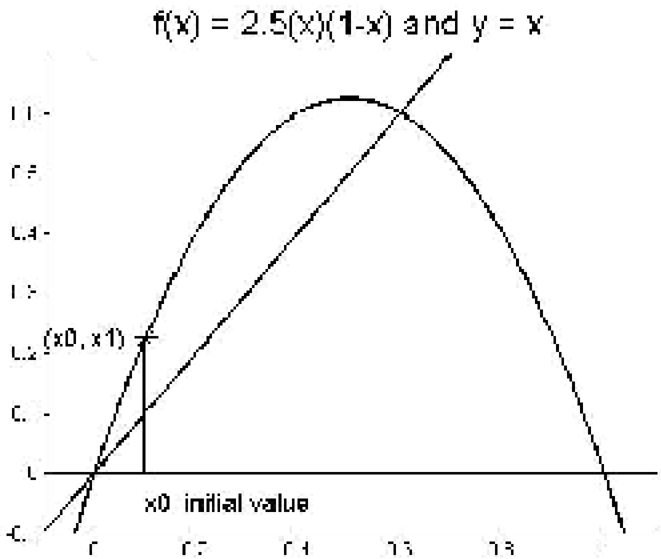
The points where the graph of the function $f(x) = 2.5x(1-x)$ intersects $y = x$ are the fixed points. The graph below shows where these intersections occur.



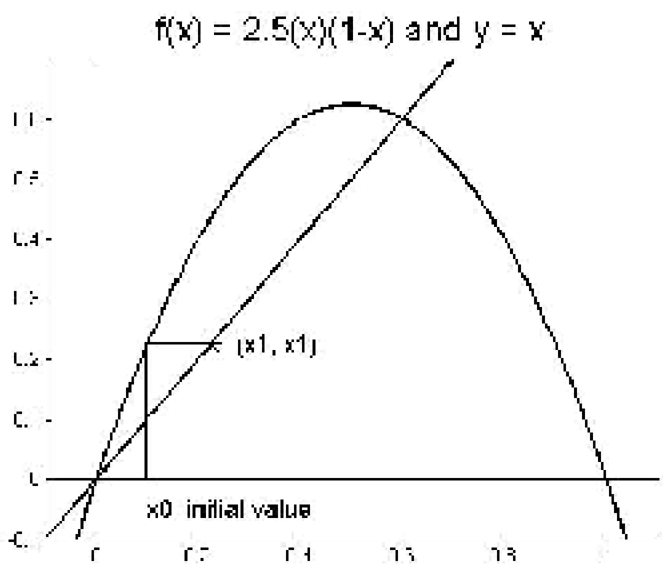
The fixed points of the logistic map are important because they relate to the orbit of an initial value x_0. These orbits will be produced graphically by using the cobweb method.

Draw the graph of $f(x) = 2.5x(1-x)$ and the graph of $y = x$. Mark the initial value $x_0 = 0.1$ on the x-axis. Draw a vertical line from x_0 to the function. This point, where the vertical line touches the function, is $(x_0, f(x_0)) = (x_0, x_1) = (0.1, 0.225)$.

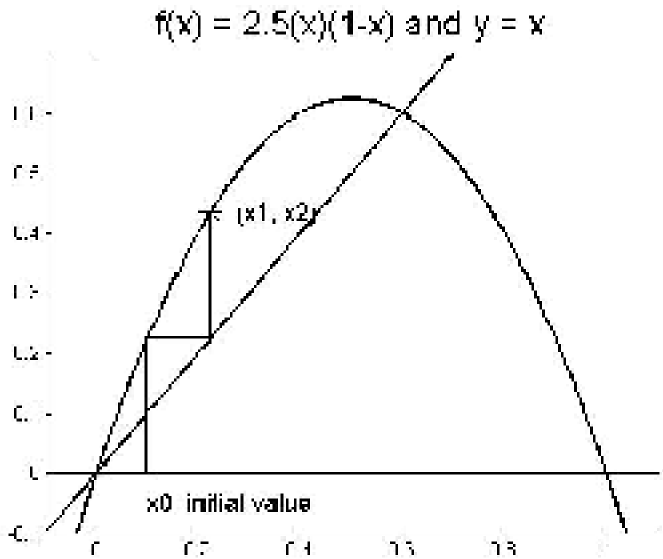
This graph shows the beginning of the cobweb with $x_0 = 0.1$.



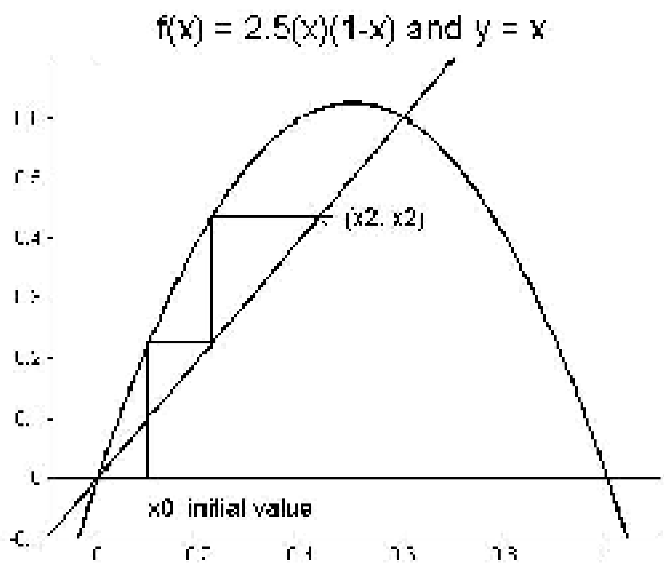
Next, draw a horizontal line from this point, (x_0, x_1), to the line $y = x$. This point is (x_1, x_1).



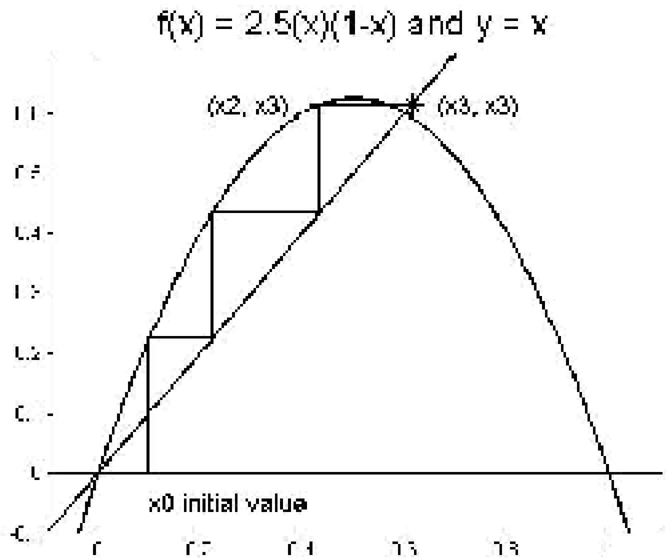
Then draw a vertical line from this point to the function; $(x_1, f(x_1)) = (x_1, x_2) = (0.225, 0.4359375)$.



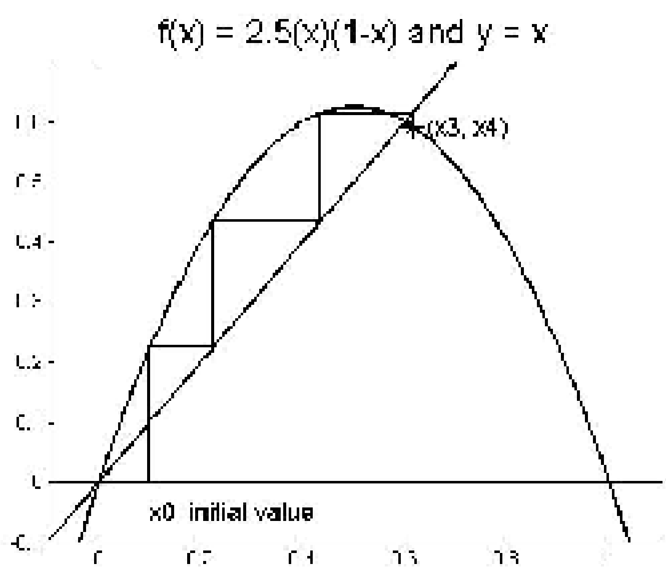
Next draw a horizontal line from the point (x_1, x_2) to the line $y = x$. This gives the point (x_2, x_2).



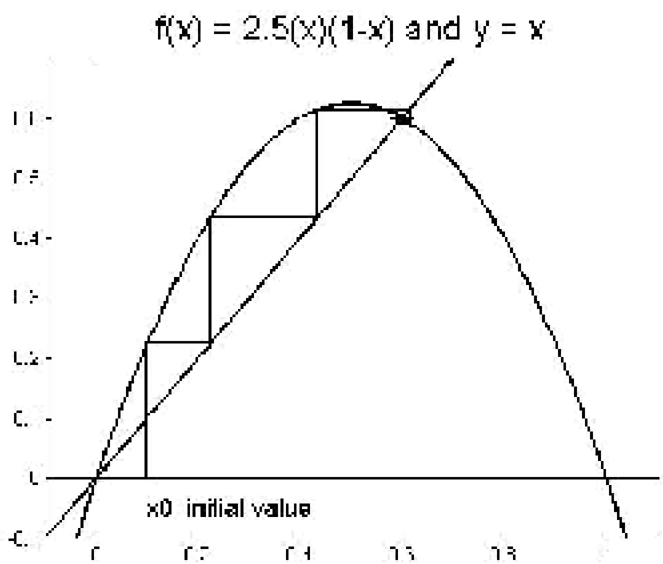
Then draw a vertical line from this point to the function; $(x_2, f(x_2)) = (x_2, x_3) = (0.4359375, .6147399902)$. Draw a horizontal line from this point to the line $y = x$, giving the point (x_3, x_3).



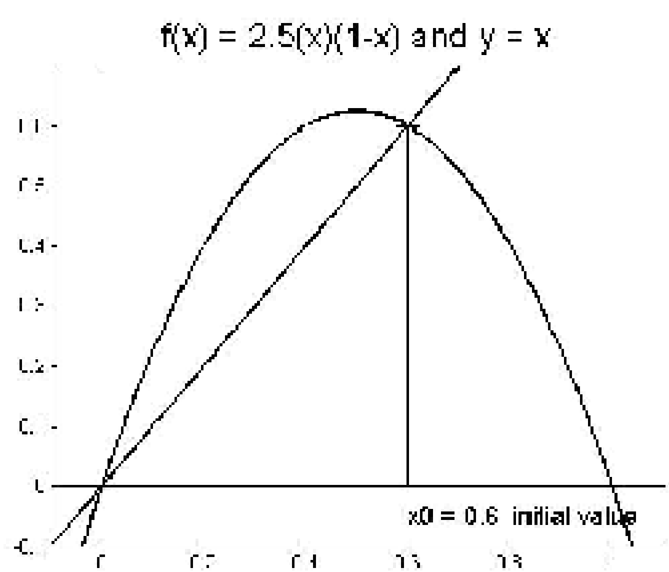
Next draw a vertical line from (x_3, x_3) to the function; $(x_3, f(x_3) = (x_3, x_4) = (0.6147399902, 0.5920868366)$.



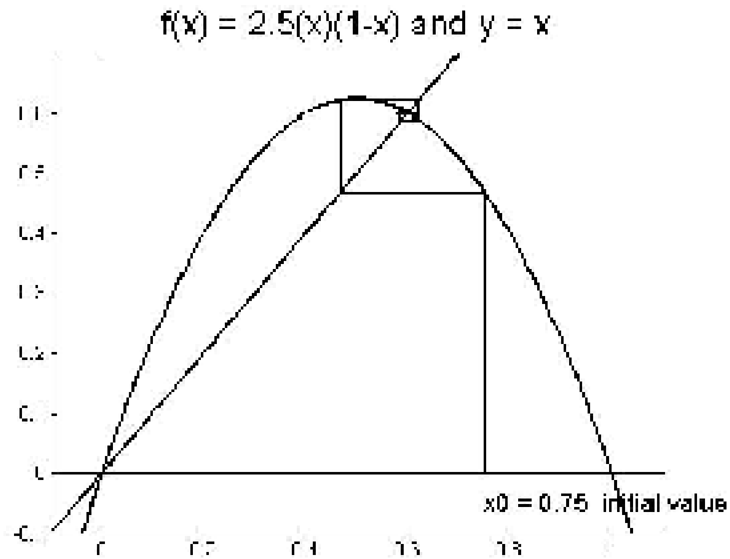
Draw a horizontal line from this point to the line y = x to get (x_4, x_4). Continue this process until it can be seen that the cobweb continues to go to a certain point. This point is the point to which the orbit of x_0 = 0.1 converges. The graph below shows x_0 through x_{20} and with so many iterations, it becomes obvious that the orbit of x_0 = 0.1 converges to the fixed point x = 0.6.



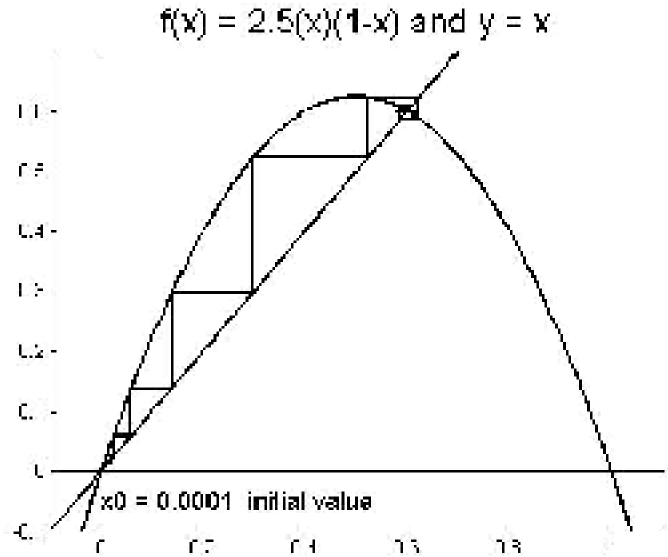
Using the cobweb method with the initial value at x_0 = 0.6, we confirm that x_0 = 0.6 is the fixed point, as displayed below.



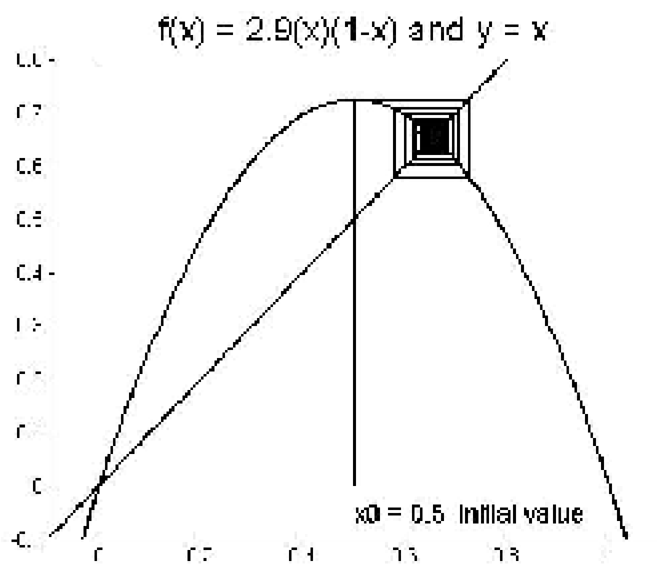
Using an initial value, $x_0 = 0.75$, taken from the right side of the fixed point, we see that the orbit is still attracted to the fixed point x = 0.6.



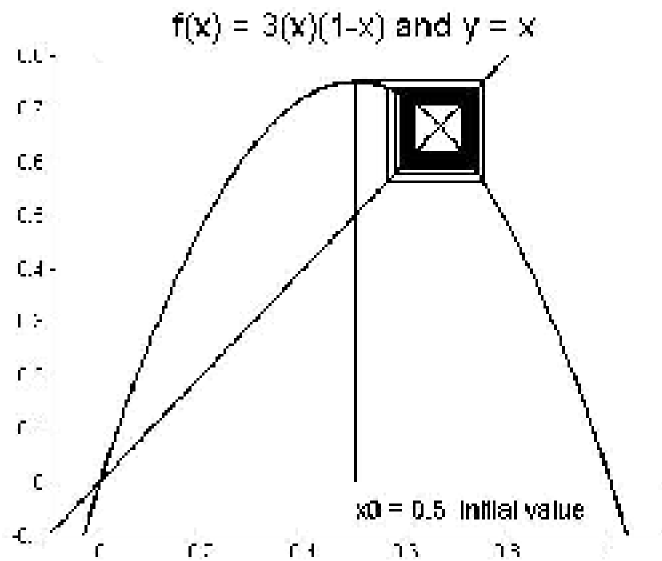
The cobweb iterations of the function f(x) = 2.5x(1-x) for a variety of initial values show how the orbits will always go toward the fixed point (x = 0.6). The only exception to this is for $x_0 = 0$. The fixed point in this case is x = 0. The fixed point x = 1 - (1/λ) is called an attracting fixed point because the orbits tend toward this point. The fixed point x = 0 is called a repelling fixed point because the orbits go away from this point except for $x_0 = 0$. The cobweb diagram for the initial value $x_0 = .0001$ shows that the fixed point x = 0 is repelling and the fixed point x = 0.6 is attracting as displayed below.



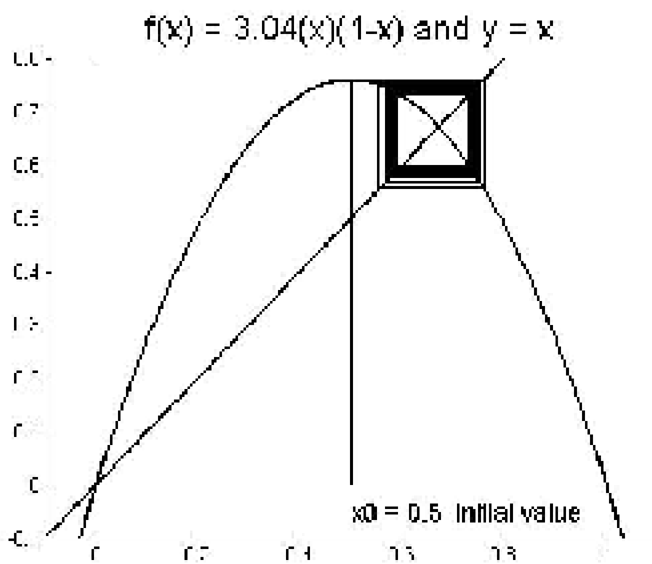
Consider the logistic map with λ = 2.9. Shown below is the cobweb for the function f(x) = 2.9x(1-x) with initial value $x_0 = 0.5$. The orbit of $x_0 = 0.5$ converges to x = (19/29) ≈ .6552.



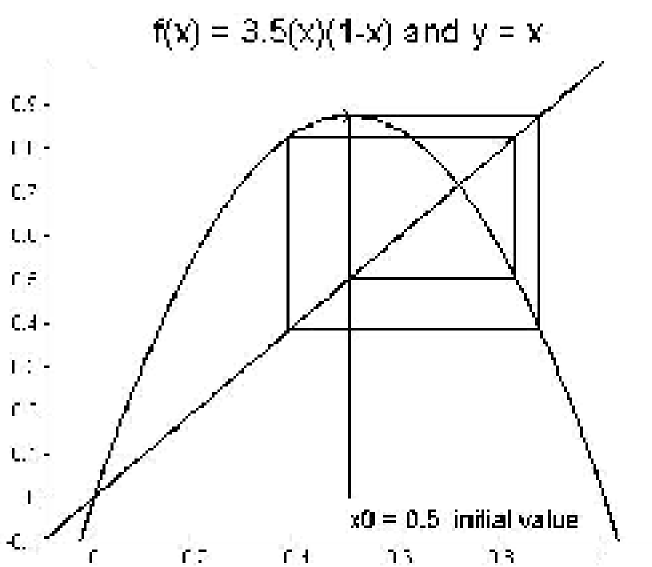
The different values taken for λ cause various qualitative types of behavior to occur. For $1 < \lambda \leq 3$ the fixed points are (0,0) and (1-(1/ λ), 1-(1/ λ)) with (0,0) repelling and (1-(1/ λ), 1-(1/ λ)) attracting. The cobweb diagram for λ = 3 is shown below.



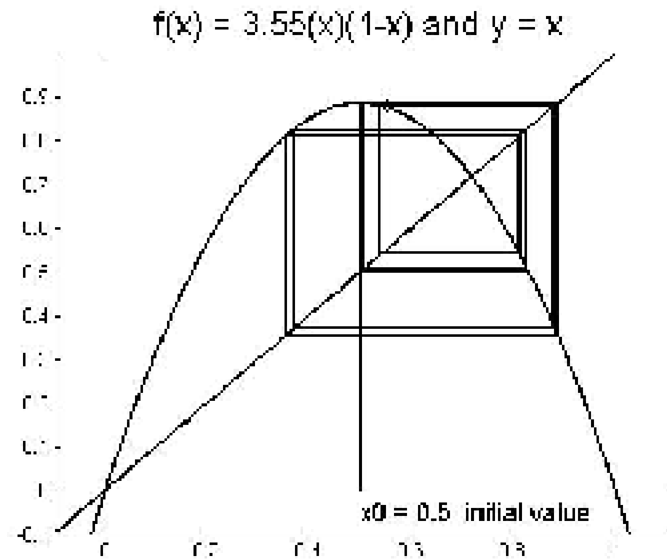
When λ = 3, the orbits converge very slowly to the fixed point ((2/3), (2/3)). After λ = 3, the fixed point x = 1 – (1/λ) will no longer be attracting, but instead will be repelling. There is a loss of stability and a 2-cycle is born and becomes the attractor. Thus λ = 3 is called a bifurcation point. When $3 < \lambda \leq 1 + \text{sqrt}(6) \approx 3.4495$, the points $x_1 = (1 + \lambda + \text{sqrt}(\lambda^2 - 2\lambda - 3)) / 2\lambda$ and $x_2 = (1 + \lambda - \text{sqrt}(\lambda^2 - 2\lambda - 3)) / 2\lambda$ form the 2-cycle for the map f and each point is a fixed point for the iterated map f^2. (Rasband, 1991, p.22). Consider an example with λ = 3.04. The 2-cycle for the orbits is $x_1 = (1 + 3.04 + \text{sqrt}(3.04^2 – 2(3.04) -3)) / 2(3.04) \approx 0.730591287\ldots$ and $x_2 = (1 + 3.04 - \text{sqrt}(3.04^2 – 2(3.04) -3)) / 2(3.04) \approx 0.5935521433\ldots$ λ= 3.04 yields a cobweb with stable cycle of period 2.



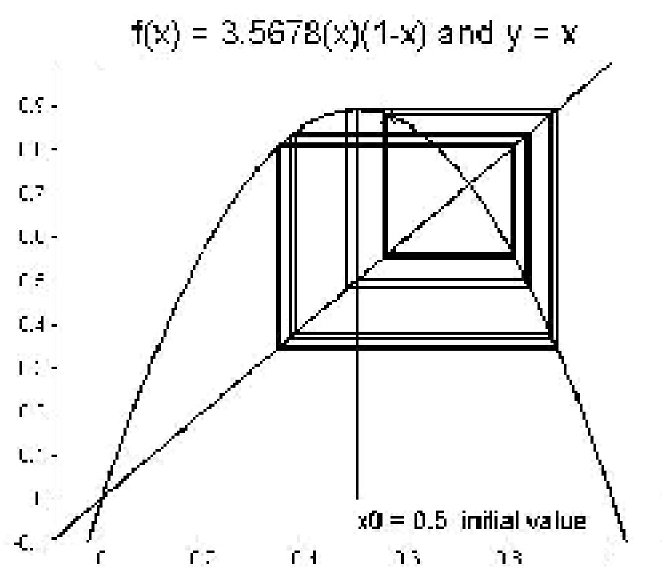
The second bifurcation occurs at λ = 1 + sqrt(6) ≈ 3.4495. For λ between approximately 3.4495 and approximately 3.5441, there are stable cycles of period 4. λ= 3.5 yields a cobweb of period 4, as shown below.



Period 8 occurs at the third bifurcation λ = 3.5441. Values ranging from approximately 3.5441 and approximately 3.5644 for λ produce cycles with eight points. λ = 3.55 yields period 8.



The range for values of λ that give a stable cycle of period 16 is between approximately 3.5644 and approximately 3.5688. λ= 3.5678 yields period 16.



There is an infinite sequence of values of λ where bifurcation occurs. The first eight are

$\lambda_1 = 3.0$,

$\lambda_2 = 3.449490\ldots$,

$\lambda_3 = 3.544090\ldots$,

$\lambda_4 = 3.564407\ldots$,

$\lambda_5 = 3.568759\ldots$,

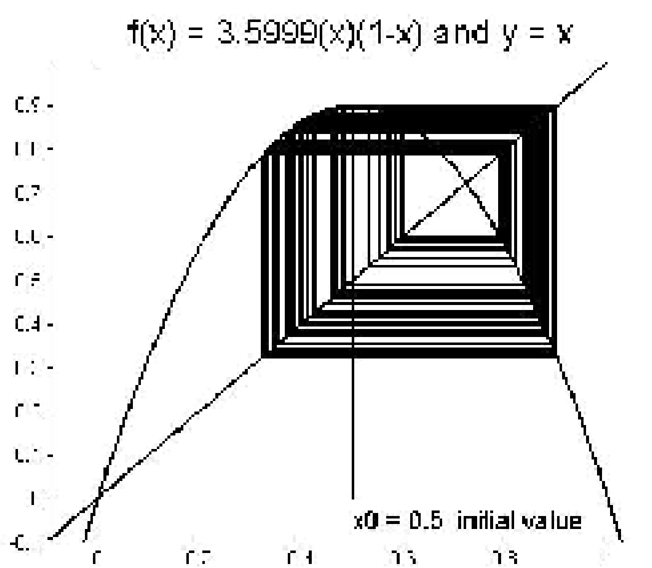
$\lambda_6 = 3.569692\ldots$,

$\lambda_7 = 3.569891\ldots$,

and $\lambda_8 = 3.569934\ldots$

(Rasband, 1991, p.23)

It is obvious that the values of λ where the bifurcations occur become closer to each other and converge rapidly to a value denoted by $\lambda_\infty = 3.5699456\ldots$. $\lambda = 3.5999$ yields chaotic behavior.



A summary of the range of values for the first seven periods is listed below:

- For $3.000000 < \lambda < 3.449490$, there is a stable cycle of period 2.
- For $3.449490 < \lambda < 3.544090$, there is a stable cycle of period 4.
- For $3.544090 < \lambda < 3.564407$, there is a stable cycle of period 8.
- For $3.564407 < \lambda < 3.568759$, there is a stable cycle of period 16.
- For $3.568759 < \lambda < 3.569692$, there is a stable cycle of period 32.
- For $3.569692 < \lambda < 3.569891$, there is a stable cycle of period 64.
- For $3.569891 < \lambda < 3.569934$, there is a stable cycle of period 128.

(Kaplan, D., & Glass, L., 1995, p.30)

The differences between these numbers where the periods double get smaller and smaller. Let Δ_n equal the difference in range values for each period, where n represents the period size. For example, Δ_2 would equal 3.4495 – 3.0000, the difference of the values of its period.

$$\Delta_2 = 3.449490 - 3.000000 = .44949.$$

$$\Delta_4 = 3.544090 - 3.449490 = .0946.$$

$$\Delta_8 = 3.564407 - 3.544090 = .020317.$$

$$\Delta_{16} = 3.568759 - 3.564407 = .004352.$$

$$\Delta_{32} = 3.569692 - 3.568759 = .000933.$$

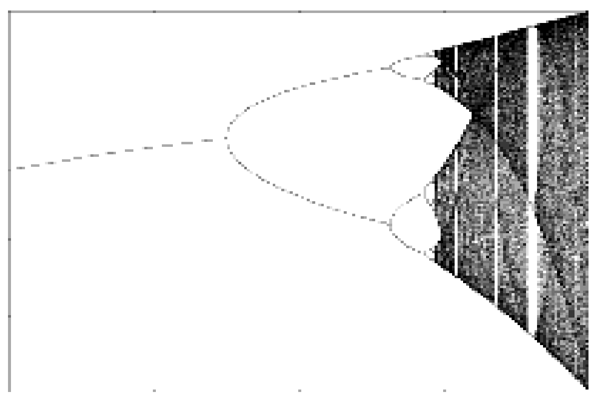
$$\Delta_{64} = 3.569891 - 3.569692 = .000199.$$

$$\Delta_{128} = 3.569934 - 3.569891 = .0000037.$$

The mathematician Mitchell J. Feigenbaum found that the limit of the quotient of successive periods is 4.6692…(Feigenbaum's constant).

$$\lim n \rightarrow \infty \ \Delta_n / \Delta_{2n} = 4.6692\ldots$$

The Feigenbaum constant can be used to approximate λ_∞ at 3.5699456… (Rasband, 1991, p.23). In order to visually show what is happening at the bifurcation points, a diagram is used. The diagram below illustrates the points at which the behavior of the system changes. The horizontal axis gives the values of λ between 1 and 4. The vertical axis gives the values to which the orbits converge or cycle. From $\lambda = 1$ to $\lambda = 3$, the orbits converge to attracting fixed points. It is at $\lambda = 3$ that the first split in the diagram occurs. It bifurcates like a fork in the road. Between $\lambda = 3$ and $\lambda = 3.5699456\ldots$ the orbits cycle between two numbers. This period-doubling bifurcation continues until the point $\lambda_\infty = 3.5699456\ldots$ is reached in the diagram. That is where chaos appears.



From $\lambda_\infty = 3.5699456\ldots$ to $\lambda = 4$, there are spaces in the chaotic behavior that appear as white strips, as seen above. These spaces indicate stable cycles. In the following section, a description of Chaos theory and the qualities that are required for a system to be Chaotic will be discussed using the logistics map.

The logistics map is only one example of a difference equation that can generate a system that exhibits Chaos. It is a good example because it is easy to see chaotic behavior. The graph below shows **sensitivity to initial conditions,** which is one of the characteristics in determining if a system is Chaotic. In this logistics model, the lower series started from an initial population of 0.202701, which was only slightly greater than the upper series of 0.2027. As displayed below, the two graphs do not show any resemblance to each other after sixteen iterations.

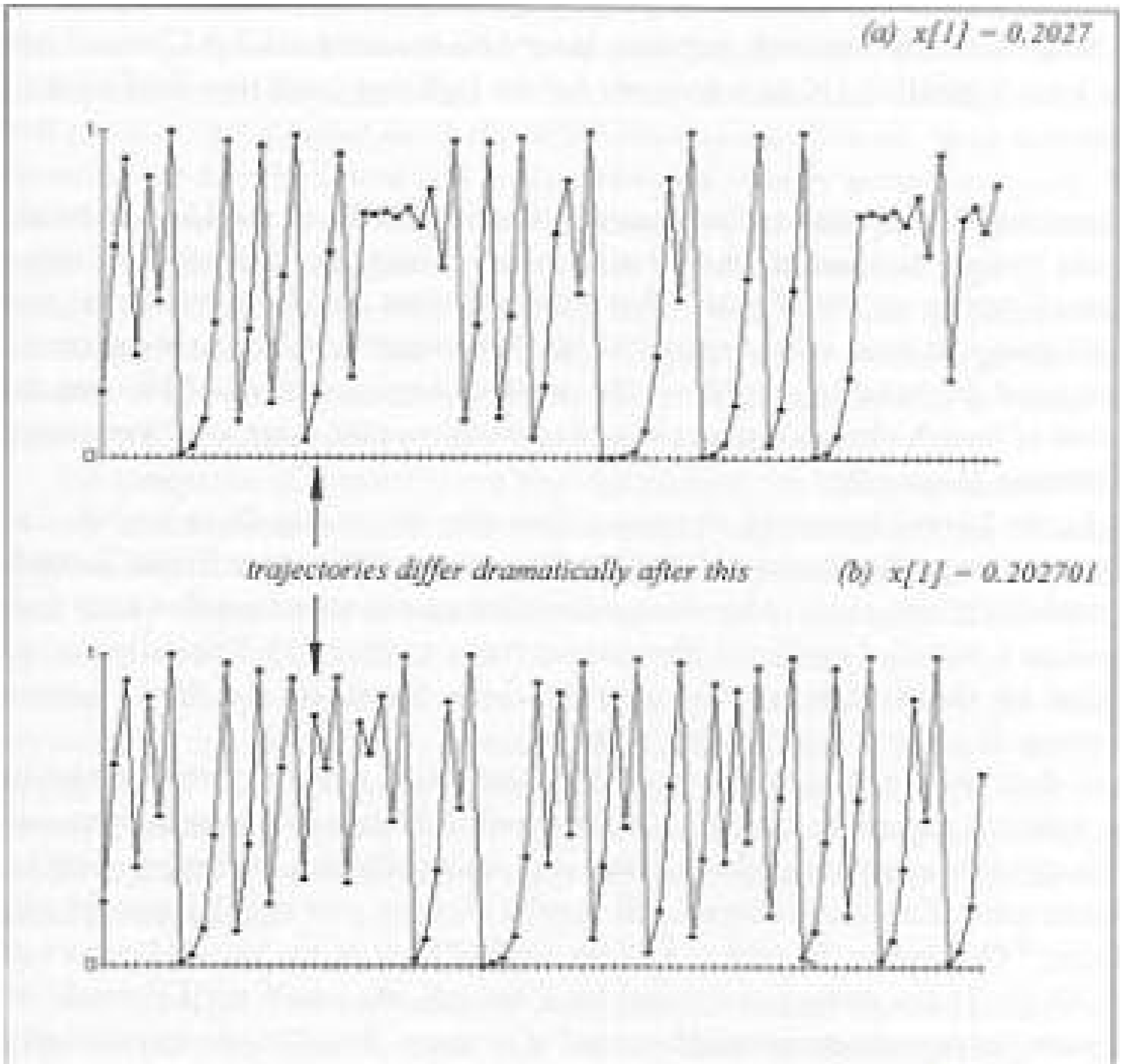


Figure 10. Chaotic Trajectories of the Iterated Logistic Map.[19]

(http://www.nwc.navy.mil/PRESS/npapers/np10/np10.pdf#search=%22Chaos%20Theory%20%26%20Glenn%20James%22 retrieved on 02/07/07)

For a system to be **deterministic**, there must be enough known about the components of the system to determine the future of the system. In other words, each state of the system depends on the state before it. The animal system described by the logistic function is deterministic because the size of the population over time is completely determined by the function and the initial size of the population.

In order for a system to display Chaos, it must be generated by **nonlinear equations**. Nonlinear equations include exponents, trigonometric functions, logarithms and other nonlinear operations in their definitions, in contrast to linear equations that are defined with only the operations of addition, subtraction, multiplication and division. Since the solutions of linear systems are known, it is impossible for a linear system to become chaotic (James, 1996, p. 29). As mentioned earlier, the logistics equation is $f(x) = \lambda x - \lambda x^2$. As one can see, the term x^2 contains an exponent. Therefore, this equation is nonlinear. We saw above that the equation $f(x) = 3.5999x - 3.5999x^2$ creates a system that acts Chaotically. So, from this example, we can see that a nonlinear equation can cause a system to act Chaotically.

Another characteristic that a system must have to act Chaotically is that it has to be **bounded**. "Bounded means that on successive iterations, the state stays in a finite range and does not approach $\pm\infty$" (Kaplan, D. et al., 1995, p.27). For example, looking at the cobweb iteration diagram that yields chaotic behavior when $\lambda = 3.5999$, it is evident that the diagram is bounded. The initial value here was $x_0 = 0.5$ and was in the range $0 \leq x_0 \leq 1$. Since this is a bounded function, all of the future iterations of the function will also fall in this range.

When observing a system, it is important to keep all of these characteristics in mind. If any one of these characteristics is missing from the system, then it is not Chaotic. A final important trait of Chaotic systems is that they are **not periodic**. In other words, the same thing does not happen over and over. An example of a periodic system would be the one created by our equation $f(x) = 3.04x(1-x)$, whereas an example of a non-periodic system would be the one created by our equation $f(x) = 3.5999x(1-x)$.

In her paper, Chaos Theory for the Practical Military Mind, Maj. Susan E. Durham states that "there are countless, real, physical systems *upon which military lives and missions depend that are Chaotic systems*" (p. 34). In the next section, some of these systems will be discussed. Turbulent air flow affects many military systems. This Chaotic turbulence has an effect on military objects such as aircraft, missiles, and paratroopers jumping out of planes. Turbulence that affects airplanes that we travel in everyday is the same turbulence that military personnel have to handle with their aircraft.

Atmospheric turbulence also affects the military's detection of objects in space. "Construction of adaptive optics sensors that take into account turbulence through the atmosphere have been one of the most significant contributions to the military's Space Object Identification mission in decades" (Durham, 1997, p. 35). This Chaotic atmosphere also affects lasers that travel through it.

In addition to appearing in the atmosphere, turbulence shows up in other military objects as well. These include missiles, radio signals, and lasers themselves. Turbulence appears in the mixing of fuels in any liquid-propelled engine such as missiles. This flow also appears in water pipes that the military builds. As mentioned above, lasers can be affected by the turbulent atmosphere. In addition to this, the actual laser is affected by

Chaotic fluctuations.

Another part of the military that exhibits Chaos is the radio signals that military forces use to send information to one another. These radio signals are for communication between friendly forces only. To prevent enemy forces from eavesdropping, signals designed to hide in background noise are transmitted. Durham suggests that the kinds of signals produced by 'chaotic oscillators' may be ideal for this. The goal is for the radio messages to sound as close to noise as possible so that enemy forces will not be able to detect their communications (Durham, 1997, p. 36).

In addition to showing up in physical systems used by the military, Chaos theory is beginning to be applied to non-physical systems as well. The way that it is being used in the military is through the philosophy of warfare. It is also being applied by military strategists to issues in political science and international relationships (Durham, 1997, p. 38). Chaos theory is starting to be applied to political issues in the military as well. "Alan Saperstein is logically and systematically investigating the applicability of the Chaos Theory to topics of military interest ranging from 'Cold War' arms-race scenarios to issues of Post-Cold War, world wide and regional political stability" (Durham, 1997, p. 40).

In order to understand how the military can deal with these systems affected by Chaos, we have to first comprehend Chaos control. Chaos control is possible because Chaotic systems can be made stable. "Three basic approaches have been demonstrated for Chaos control: regular periodic disturbances, proportional inputs based on real-time feedback, and trajectory 'steering' based on models or approximations of the dynamics on an attractor" (James, 1996, p. 68).

Regular periodic disturbances make Chaos control possible because when these disturbances act on a Chaotic system, they excite natural frequencies in the system. In a Chaotic system, infinitely many different periodic behaviors are possible. Disturbing a system into periodic behavior brings order to a Chaotic system.

Proportional inputs based on real-time feedback also make Chaos control possible. This control method "requires real-time measurements of the system's output in order to determine how far to adjust the selected control parameter" (James, 1996, p. 68). An example of how this control parameter works is when someone tries to balance a long stick on the palm of his/her hand. He/she must move his/her hand slightly to keep the stick from falling from the upright position. This method produces a stable output that is achieved intentionally.

The last method, trajectory 'steering' based on models or approximations of the dynamics on an attractor, is more complicated and complex than the first two. This method was developed at the Massachusetts Institute of Technology (MIT) and it is based on approximations to the dynamics on a system's attractor that are made through extensive calculations. After the approximations are made, they are used to adjust the system parameters to make it possible to control the system to preferred regions of phase space and therefore control Chaos (James, 1996, p. 69).

All three of these techniques are possible due to the system attractors. Locking onto one of the unstable periodic trajectories densely woven on an attractor, allows one to control Chaos (James, 1996, p. 69).

The applications of Chaos theory in the military are extensive. The next section will discuss these applications and how these applications are possible because of Chaos control. Applications of Chaos theory include using it in military communications, decoding enemy signals, searching for enemy missiles, and using it in assembly lines. Part of understanding how to use Chaos in the military comes from knowing how Chaos can be controlled.

One section of the military that is beginning to look at using Chaos is military communications. "The high volume and speed of communication through computer networks include the best ingredients of a recipe for Chaos: modular processes undergoing endless iteration; frequent feedback in communications 'handshaking'; and frequencies (on many scales) faster than the time it takes most systems to recover between 'events' (messages, transmissions, and back-ups)" (James, 1996, p. 51). Chaos may be used to either secure friendly communications, as mentioned above, or break enemy communication codes. The military uses communication for everything it does and it is important to keep these communications as secure as possible. Understanding how to detect Chaos in a system and looking at the possibility of using Chaos to decode enemy signals can be essential for military personnel, especially when combating terrorism.

Another possible military application for Chaos theory is using it to search for enemy missiles that terrorists may use to carry out their acts of violence. The more terrorist attacks that can be prevented, the more peaceful the world will be.

Chaos can also be controlled in the military when dealing with assembly lines. The Chaotic dynamics in automatic production lines such as robotic systems for aircraft stripping and painting, and automated search algorithms for hostile missiles or ground forces can be managed because of their Chaotic behavior (James, 1996, p. 51).

By comprehending how to use Chaos to control military systems, military personnel will be better equipped to prevent terrorist attacks that have been growing over the past few years around the world. This fairly new mathematical concept is still being explored and it is expanding in its applications for the military. It is important that military personnel understand the vast use of Chaos theory in the military and understand the basics of Chaos theory. The fundamentals of Chaos theory, military systems that are Chaotic, Chaos in non-physical systems, Chaos control, and the applications of Chaos theory for the military show the beginning process of how to use Chaos theory in the military. By understanding Chaos theory, military workers will be at an advantage in fighting techniques, information technologies and strategic thought, which in turn will create a more harmonious world with less terrorism.

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